Minimizing Flight Technical Error For Arrivals At Busy Airports – Why Would We Want To Do Such A Thing?

Timothy Hall*

Abstract
To advance aviation safety throughout the National Airspace System, the concept of Flight Technical Error (FTE) is defined to measure the extent to which an aircraft's arrival path deviates from the arrival runway extended centerline. Among many other uses, the FTE measure is used to make sure the wakes from aircraft arriving on closely-spaced parallel runways do not endanger each other's safe operation. This paper documents a least-squares best-fit FTE stochastic formulation based on two-dimensional straight-line arrival track data.

Key Words: Flight Technical Error, Least-Squares Estimation, Stochastic Models

1. Introduction

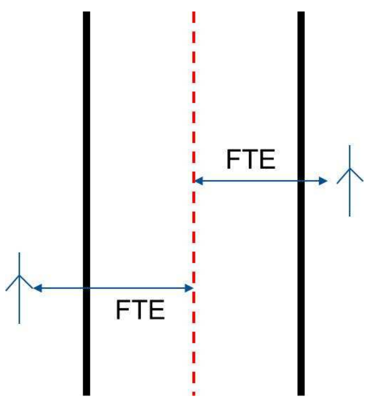

Figure 1: FTE Relative To Runway Centerline

Flight Technical Error (FTE) is how far off the centerline of the arrival runway an aircraft wanders when trying to land (see Figure 1). It measures the extent to which an aircraft's arrival path deviates from the actual arrival runway extended centerline by first assessing its deviations from its perceived centerline through ground-based, space-based, and on-board surveillance systems.

The FTE varies by distance from the touchdown point (where the wheels first come into contact with the runway – which itself may be considered a function of time). Such deviations have several sources acting simultaneously yet not necessarily independently, and it is the combined effect of all these sources that result in an observed or perceived FTE value. Among these influences are ...

- *Environmental*
  - Cross winds
  - Limited visibility
  - Flight crew disorientation
- *Operational*
  - Need to avoid obstructions
  - By instruction from Air Traffic Control
  - Offset approaches (at oblique angles to the runway itself)
- *Sensors*
  - Non-optimal sensor placement
  - Misaligned/out-of-service navigation aides
  - Dilution of precision (from being too far away from the sensors)
- *Calculations*
  - Data corruption during transmission

*PQI Consulting, P. O. Box 425616, Cambridge, MA, USA, 02142-0012

– Limited accuracy
– Uncoordinated communications
– Processor latency

Airborne aircraft positions (and therefore FTE values) are (currently) assessed by ground-based surveillance systems (such as airport-based radars) and through land-based and space-based local and global navigation "triangulation" systems such as multilateration and GPS. However, each of these devices have jointly dependent measurement error distributions that combine (in some way) to form a singe measurement error distribution whose components are unobservable in practice. This must be considered in combination with the *actual* error, i.e., the physical distance away from the centerline, to form a "total" observed distance from the centerline – actual plus observational – and this total is called the FTE.

This memorandum provides the analytical and calculation methods for finding the observed or "perceived" FTE (that observed through an aircraft's on-board avionics and through ground-based and space-based surveillance systems). This perceived FTE may then be adjusted based on the actual arrival runway extended centerline to form an actual FTE value. These analytical methods are simple enough, and straightforward enough, to be available for use in embedded systems, such as mobile tracking systems or decision-support control systems and application-specific integrated circuits.

2. Why Do We Care About FTE?

The most important reason anyone cares about effectively and efficiently measuring the FTE is **aviation safety**. It is critically important for the flight crew and air traffic controllers to know exactly where an aircraft is at all times to keep it safely separated from all other aircraft operating nearby. Furthermore, at busy airports, where multiple aircraft may be landing at the same time on closely-spaced parallel runways, one arriving aircraft's **wake turbulence** (the potentially disastrous disruption of airflow behind and below an aircraft due to its movement through the air) must not be allowed to interfere with the safe operation of another nearby arriving aircraft. And this situational awareness also makes aborted landings, go-arounds, and holding patterns less likely to be necessary.

Secondary to safety, airport and city management wants to maximum **runway throughput** (the number of aircraft arriving and departing on a single runway) and minimize **runway occupancy** (the time an aircraft remains on the active runway after landing) by spacing aircraft as closely together as may safely be allowed, and to have all aircraft land as close to the centerline of the runway as possible (so it does not have to waste time by missing high-speed turn-off maneuvers to get to its gate as soon as possible). This may be accomplished in large part by minimizing the FTE.

Finally, airline operations want to reduce fuel costs, provide critical aircraft maintenance, and maximize crew availability by minimizing the distance (and therefore time) an arriving aircraft remains in the air, i.e., by minimizing the FTE.

3. How Do We Measure FTE?

Considering all the sources of error when measuring FTE (from position, sensor, and calculation perspectives), an optimal measure of FTE would orthogonalize (to the greatest extent possible) all individual sources of error. While this might be possible in a controlled experiment with specially equipped aircraft, in day-to-day operations, a practical alternative is to collectively consider all sensor and calculation sources of error as a single factor, and to consider the position error (the actual distance off the arrival runway extended centerline) as the only analysis variable under consideration. This is the approach taken here.

3.1 A Complicating Consideration

It would greatly simplify any FTE analysis if the sources of sensor error were identical – however, they are not. Each aircraft senses its position (by whatever means) slightly differently than the next. This means Aircraft #1 may think it is 100 feet off the centerline when it is 1 nautical mile from its touchdown point, while Aircraft #2, in the identical position on the same arrival track thinks it is 110 feet off the centerline. This comes from the signal processing error that accompanies every avionics system, including GPS, and the calibration, wear, lack of maintenance, and random errors that vary (sometimes tremendously) from aircraft to aircraft. So what might be done to salvage a reasonable FTE calculation?

3.2 A Possible Solution

Referring to Figure 2, when an aircraft is flying its landing approach, it senses a "perceived centerline" based on its on-board avionics and ground-based and space-based surveillance aides. Therefore, it has a "perceived FTE" as the distance it is off from its perceived extended runway centerline. In reality, the actual FTE might be different (usually greater, but

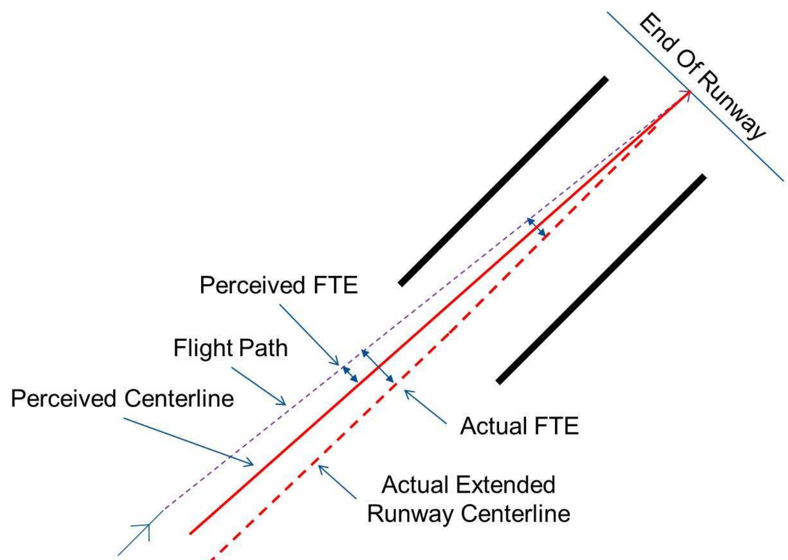

Figure 2: Perceived Versus Actual Arrival Runway Extended Centerline

sometimes smaller). Since this variability applies to every aircraft, some of them sensing the centerline to be closer than in reality and some of them sensing it to be further away, we might take the average of these perceived FTE values to be the actual FTE value. This simplification relies on the fact that avionics and surveillance sensor errors demonstrably follow a normal (Gaussian) distribution with mean 0 and non-zero variance. This is the approach taken here.

In this respect, referring to Figure 3, if the end of the runway is considered a fixed "Pivot Point" for the perceived centerline for each aircraft, and the "Free Point" is another point along that perceived centerline, then the line from the Pivot Point to the Free Point that minimizes the sum of squared perceived FTE values across a large number of arrivals (the

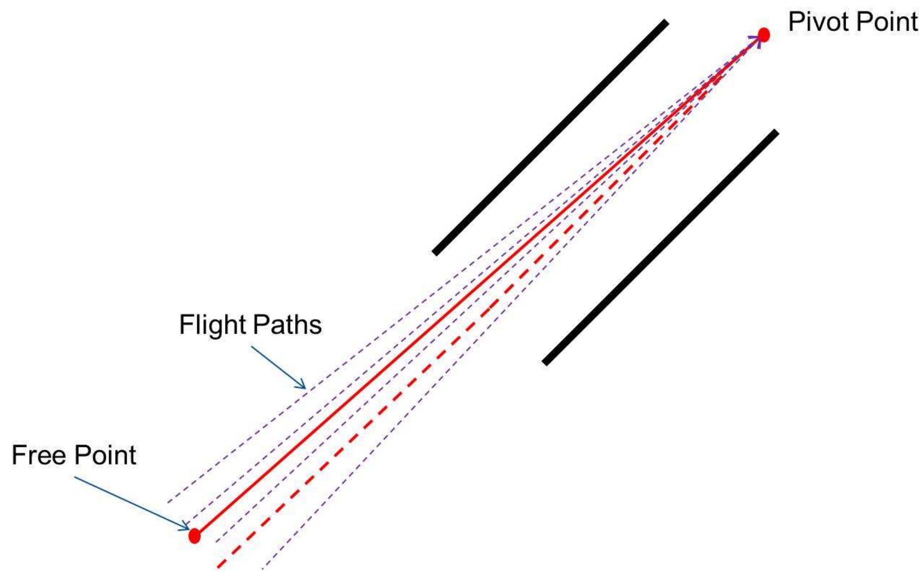

Figure 3: Estimation Of Best-Fit Perceived Arrival Runway Extended Centerline

"Best Fit" FTE) will closely approximate the typical actual FTE found during operations. Indeed, due to non-linear flight paths, the least-squares Free Points must be assessed at various distances from the Pivot Point, then combined (in some way) to produce a single perceived centerline for a final, actual FTE calculation.

3.3 Analytical Challenges

While this approach to calculating an actual FTE (and its distribution) appears to be straightforward, it does present several analytical challenges. The most pressing ones are listed here with the solutions assumed to have already occurred in the following analytical development.

- *Issue*: Characterizing the distance from an aircraft to a parameterized straight line using Cartesian geometry does not apply to latitude/longitude (GPS) coordinates.
  - *Solution*: A distance-preserving stereographic projection must first be made to a rectilinear coordinate system.
  - An alternative is to use Great Circle geodetic distances (usually expressed in terms of arc length) that are converted into a linear distance system (such as nautical miles).
- *Issue*: Combining least-squares Free Points at various distances into a single position is not obvious.
  - *Solution*: Consider all the perceived FTE values at a random set of distances as a single dataset.
  - Some applications avoid the whole issue by reporting the set of slopes/intercepts of the perceived centerline at various distances from the Pivot Point.
- *Issue*: Data integrity, especially observational variance, is not well-controlled.
  - *Solution*: Consider all observed data as the result of extensive data integrity "scrubbing," including the replacement of missing, corrupt, and nonsensical data with "corrected" values from other sources.
  - Sensor data are engineered for flight navigation, air traffic surveillance, and low-altitude situational awareness purposes, not for quality control considera-

tions and after-the-fact operations research analyses.

4. Calculation Of "Best Fit" FTE From Track Data

The following analytical methods provide a relatively easy formula (solving a fourth-degree polynomial) for calculating "best fit" FTE values for a given set of track data.

A commonly used measure of "best fit" is finding the minimum sum of squared perceived FTE distances for all track data points. This is the approach used in finding regression and correlation coefficients. Such a calculation requires mathematically parameterizing the perceived centerline in terms of a single parameter (the slope of the line joining the [variable] Free and [fixed] Pivot Points), then finding the value of that parameter that minimizes the sum of squared perceived FTE distances.

4.1 Preliminary Development

Let (x_{t_i}, y_{t_i}) be the aircraft's position at time $T = t_i$ after translation so that the Pivot Point (reputedly aligned with the runway centerline) is the origin. Using the nomenclature of the Appendix, with $(x_1, y_1) = (0, 0)$, the square of the distance from $(x_0, y_0) = (x_{t_i}, y_{t_i})$ to the line $y = mx$ is

$$d_{t_i}^2 = (By_{t_i} + (C-1)\,x_{t_i})^2 + ((A-1)\,y_{t_i} + Bx_{t_i})^2$$

where

$$A = \frac{m^2}{m^2+1},\ B = \frac{m}{m^2+1},\ \text{and } C = \frac{1}{m^2+1}$$

We want to find the value of m that minimizes the sum of all $d_{t_i}^2$ values based *only* on the track data. Since

$$\begin{pmatrix} y_{t_i} & x_{t_i} & 0 \\ 0 & y_{t_i} & x_{t_i} \end{pmatrix}_{2\times 3} \begin{pmatrix} A-1 \\ B \\ C-1 \end{pmatrix}_{3\times 1} = \begin{pmatrix} (A-1)\,y_{t_i} + Bx_{t_i} \\ By_{t_i} + (C-1)\,x_{t_i} \end{pmatrix}_{2\times 1}$$

then

$$\begin{aligned} d_{t_i}^2 &= \begin{pmatrix} A-1 \\ B \\ C-1 \end{pmatrix}^T \begin{pmatrix} y_{t_i} & x_{t_i} & 0 \\ 0 & y_{t_i} & x_{t_i} \end{pmatrix}^T \begin{pmatrix} y_{t_i} & x_{t_i} & 0 \\ 0 & y_{t_i} & x_{t_i} \end{pmatrix} \begin{pmatrix} A-1 \\ B \\ C-1 \end{pmatrix} \\ &= \begin{pmatrix} A-1 \\ B \\ C-1 \end{pmatrix}^T \begin{pmatrix} y_{t_i}^2 & x_{t_i}y_{t_i} & 0 \\ x_{t_i}y_{t_i} & x_{t_i}^2 + y_{t_i}^2 & x_{t_i}y_{t_i} \\ 0 & x_{t_i}y_{t_i} & x_{t_i}^2 \end{pmatrix} \begin{pmatrix} A-1 \\ B \\ C-1 \end{pmatrix} \end{aligned}$$

Therefore, for position data at times $\{t_i\}$, $i = 1, 2, \ldots, N$, we have

$$\sum_{k=1}^{N} d_{t_k}^2 = \begin{pmatrix} A-1 \\ B \\ C-1 \end{pmatrix}^T \begin{pmatrix} K & H & 0 \\ H & G+K & H \\ 0 & H & G \end{pmatrix} \begin{pmatrix} A-1 \\ B \\ C-1 \end{pmatrix}$$

where

$$G = \sum_{k=1}^{N} x_{t_k}^2,\ H = \sum_{k=1}^{N} x_{t_k}y_{t_k},\ \text{and } K = \sum_{k=1}^{N} y_{t_k}^2$$

Since A, B, and C are all functions of m (and only m), we can write

$$\left(\sum_{k=1}^{N} d_{t_k}^2\right)(m) = \begin{pmatrix} -\frac{1}{m^2+1} \\ \frac{m}{m^2+1} \\ -\frac{m^2}{m^2+1} \end{pmatrix}^T \begin{pmatrix} K & H & 0 \\ H & G+K & H \\ 0 & H & G \end{pmatrix} \begin{pmatrix} -\frac{1}{m^2+1} \\ \frac{m}{m^2+1} \\ -\frac{m^2}{m^2+1} \end{pmatrix}$$
$$= \frac{1}{(m^2+1)^2}\left(Gm^4 - 2Hm^3 + (G+K)\,m^2 - 2Hm + K\right)$$

Then

$$\frac{d}{dm}\left(\sum_{k=1}^{N} d_{t_k}^2\right)(m) = \frac{d}{dm}\left(\frac{Gm^4 - 2Hm^3 + (G+K)\,m^2 - 2Hm + K}{(m^2+1)^2}\right)$$
$$= \frac{2}{(m^2+1)^3}\left(Hm^4 + (G-K)\,m^3 + (G-K)\,m - H\right)$$

since G, H, and K are all constant with respect to m.

The value of $m = m_0$ which makes

$$\frac{d}{dm}\left(\sum_{k=1}^{N} d_{t_k}^2\right)(m_0) = 0$$

is the value of m that either maximizes or minimizes $\sum_{k=1}^{N} d_{t_k}^2$. This means the (real) solutions to

$$Hm^4 + (G-K)\,m^3 + (G-K)\,m - H = 0 \tag{1}$$

contain the maximizing and minimizing values of m (since $m^2+1 \neq 0$ for all (real) m). However, (1) is a fourth-degree polynomial in m, which means there are either four, or two, or zero real solutions (counting multiplicity). In general, the value $m = m_0$ that minimizes $\sum_{k=1}^{N} d_{t_k}^2$ means the value $m = -\frac{1}{m_0}$ maximizes $\sum_{k=1}^{N} d_{t_k}^2$, so two real and two complex values will be available regardless of the values of G, H, and K.

The implementation procedure must choose the minimizing value of m among all solutions to (1).

4.2 The Minimizing Value Of m

Suppose $H \neq 0$, i.e., the $\{x_i\}$ and $\{y_i\}$ vectors are not orthogonal. Then we may consider the zeros of

$$m^4 + \left(\frac{G-K}{H}\right)m^3 + \left(\frac{G-K}{H}\right)m - 1 = 0$$

Suppose further there were real numbers (p, q, r, s) such that

$$m^4 + \left(\frac{G-K}{H}\right)m^3 + \left(\frac{G-K}{H}\right)m - 1 = \left(m^2 + pm + q\right)\left(m^2 + rm + s\right)$$
$$= \begin{pmatrix} m^4 + (p+r)\,m^3 \\ +\,(q+s+pr)\,m^2 \\ +\,(ps+qr)\,m + qs \end{pmatrix} \tag{2}$$

where $q \neq 0$.

Let

$$s = -\frac{1}{q}$$

so that (2) becomes

$$m^4 + (p+r)\, m^3 + \left(q - \frac{1}{q} + pr\right) m^2 + \left(-\frac{p}{q} + qr\right) m - 1 = 0$$

Therefore

$$\begin{aligned} q - \frac{1}{q} + pr &= 0 \\ q^2 + prq - 1 &= 0 \\ q &= \frac{-pr \pm \sqrt{p^2r^2+4}}{2} \\ \frac{1}{q} &= \frac{pr \pm \sqrt{p^2r^2+4}}{2} \end{aligned}$$

means the form of (2) becomes

$$\begin{aligned} & m^4 + (p+r)\, m^3 + \left(\frac{-p^2r \mp p\sqrt{p^2r^2+4}}{2} + \frac{-pr^2 \pm r\sqrt{p^2r^2+4}}{2}\right) m - 1 \\ &= m^4 + (p+r)\, m^3 + \frac{1}{2}\left(-pr\,(p+r) \pm (r-p)\sqrt{p^2r^2+4}\right) m - 1 \end{aligned}$$

Then

$$\begin{aligned} p + r &= \frac{G-K}{H} \\ r &= \frac{G-K}{H} - p \end{aligned}$$

so that

$$\begin{aligned} \frac{1}{2}\left(-pr\,(p+r) + (r-p)\sqrt{p^2r^2+4}\right) &= \frac{1}{2}\left(\begin{array}{c} -p\left(\frac{G-K}{H} - p\right)\left(p + \left(\frac{G-K}{H} - p\right)\right) \\ \pm\left(\left(\frac{G-K}{H} - p\right) - p\right)\sqrt{p^2\left(\frac{G-K}{H} - p\right)^2 + 4} \end{array}\right) \\ &= -\frac{1}{2H^2}\left((K-G)\left(Hp^2 + (K-G)\,p \pm H\Lambda\right) \pm 2\Lambda H^2 p\right) \end{aligned}$$

where

$$\Lambda = \sqrt{p^2\left(p - \frac{1}{H}(G-K)\right)^2 + 4}$$

Then given (G, H, K), solving

$$-\frac{1}{2H^2}\left((K-G)\left(Hp^2 + (K-G)\,p \pm H\Lambda\right) \pm 2\Lambda H^2 p\right) = \frac{G-K}{H} \qquad (3)$$

for p also provides r and q and s through

$$\begin{aligned} r &= \frac{G-K}{H} - p \\ q &= \frac{-pr \pm \sqrt{p^2r^2+4}}{2} \\ s &= -\frac{1}{q} = \frac{-pr \mp \sqrt{p^2r^2+4}}{2} \end{aligned}$$

Note that $q \neq 0$ necessarily since

$$\sqrt{p^2r^2+4} > |pr|$$

for real p and r.

Without loss of generality, we may take $q > 0$ (so that $s < 0$), which simplifies the solution m to

$$\begin{aligned}
-\frac{1}{2H^2}\left(\left(K-G\right)\left(Hp^2+\left(K-G\right)p+H\Lambda\right)+2\Lambda H^2 p\right) &= \frac{G-K}{H} \\
r &= \frac{G-K}{H}-p \\
q &= \frac{-pr+\sqrt{p^2r^2+4}}{2} \\
s &= \frac{-pr-\sqrt{p^2r^2+4}}{2}
\end{aligned}$$

Then since $s < 0$, we necessarily have real solutions of m from

$$m = \frac{-r \pm \sqrt{r^2-4s}}{2}$$

and also from

$$m = \frac{-p \pm \sqrt{p^2-4q}}{2}$$

when $p^2 > 4q$.

The discriminant that determines whether m supplies a minimum or maximum value of $\sum_{k=1}^{N} d_{t_k}^2$ is given by

$$\Psi\left(m\right) = 4Hm^3 + 3\left(G-K\right)m^2 + G - K$$

in the sense of the concavity of $\sum_{k=1}^{N} d_{t_k}^2$ at m.

Finally, if $H = 0$, then (2) becomes

$$\left(G-K\right)m^3 + \left(G-K\right)m = 0$$

which only has $m = 0$ as a solution unless $G = K$, in which case (2) is vacuous.

4.3 A Stochastic Data Model

Suppose X were the stochastic process that produces the $\{x_{t_i}\}$ data and Y were the stochastic process that produces the $\{y_{t_i}\}$ data. If X and Y are independent and if θ is the incident angle of the arrival runway extended centerline to the x-axis, then

$$\begin{aligned}
X' &= \begin{pmatrix} \cos\left(\frac{\pi}{2}-\theta\right) \\ \sin\left(\frac{\pi}{2}-\theta\right) \end{pmatrix}^T \begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(0, \sigma_X^2\right) \\
Y' &= \begin{pmatrix} \cos\left(-\theta\right) \\ \sin\left(-\theta\right) \end{pmatrix}^T \begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left(0, \sigma_Y^2\right)
\end{aligned}$$

would represent the same normal (Gaussian) distribution stochastic nature of the x and y components, respectively, as is present in the cross-track lateral stochastic nature of the X and Y random variables.

However,

$$X' = \cos\left(\frac{\pi}{2} - \theta\right) X + \sin\left(\frac{\pi}{2} - \theta\right) Y = (\sin\theta) X + (\cos\theta) Y$$
$$Y' = \cos(-\theta) X + \sin(-\theta) Y = (\cos\theta) X - (\sin\theta) Y$$

or

$$\begin{aligned}\begin{pmatrix} X \\ Y \end{pmatrix} &= \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}^{-1} \begin{pmatrix} X' \\ Y' \end{pmatrix} \\ &= \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} \\ &= \begin{pmatrix} (\sin\theta) X' + (\cos\theta) Y' \\ (\cos\theta) X' - (\sin\theta) Y' \end{pmatrix}\end{aligned}$$

so that

$$X \sim N\left(0, \left(\sin^2\theta\right)\sigma_X^2 + \left(\cos^2\theta\right)\sigma_Y^2\right)$$
$$Y \sim N\left(0, \left(\cos^2\theta\right)\sigma_X^2 + (-\sin\theta)^2\sigma_Y^2\right)$$

In the special (yet common) case that $\sigma_X^2 = \sigma^2 = \sigma_Y^2$, then

$$X, Y \sim N\left(0, \sigma^2\right)$$

Hence, each observed $By_{t_i} + (C-1)\,x_{t_i}$ value comes from a

$$N\left(0, \left(\left(\cos^2\theta\right)\sigma_X^2 + (-\sin\theta)^2\sigma_Y^2\right)B^2 + \left(\left(\sin^2\theta\right)\sigma_X^2 + \left(\cos^2\theta\right)\sigma_Y^2\right)(C-1)^2\right)$$

distribution, and each observed $(A-1)\,y_{t_i} + Bx_{t_i}$ value comes from a

$$N\left(0, \left(\left(\cos^2\theta\right)\sigma_X^2 + (-\sin\theta)^2\sigma_Y^2\right)(A-1)^2 + \left(\left(\sin^2\theta\right)\sigma_X^2 + \left(\cos^2\theta\right)\sigma_Y^2\right)B^2\right)$$

distribution. This means each d_{t_i} observation comes from the positive square root of the sum of two squared dissimilar non-standard normal (Gaussian) distributions. In the simplified case where $\sigma_X^2 = \sigma^2 = \sigma_Y^2$, then

$$\frac{By_{t_i} + (C-1)\,x_{t_i}}{\sigma\sqrt{B^2 + (C-1)^2}} \sim N(0,1)$$
$$\frac{(A-1)\,y_{t_i} + Bx_{t_i}}{\sigma\sqrt{(A-1)^2 + B^2}} \sim N(0,1)$$

and under the further simplified case where $m^2 = 1$, so that $A = |B| = C = \frac{1}{2}$, then[1]

$$|d_t| \sim \frac{\sigma}{\sqrt{2}}\sqrt{\chi^2(2)} \tag{4}$$

[1]A random variable Y is said to have a cX distribution, for a constant $c \neq 0$, if

$$\frac{1}{c}Y \sim X$$

where the density function $f_{d_t}(u)$ of $\sqrt{\chi^2(2)}$ is given[2] by

$$f_{d_t}(z) = ze^{-\frac{1}{2}z^2} \tag{5}$$

Therefore, if the "raw" (X, Y) data were rotated by angle[3] $\frac{\pi}{4} - \text{Arctan}\, m$ so that $m = 1$, then under the assumption that $\sigma_X^2 = \sigma^2 = \sigma_Y^2$, the distribution of $|d_t|$ is given by (4) with density function given by (5). In the absence of this rotation or this coordinate variance assumption, the distribution of $|d_t|$ is the positive square root of the sum of two squared dissimilar non-standard normal (Gaussian) distributions.

5. How Do We Use FTE?

While the use of FTE information is only limited by the imagination of the analyst, the most common uses address the safe and efficient operation of aircraft arriving at, and departing from, busy airports throughout the world.

- *Wake Mitigation*
  - Controllers space aircraft apart from each other with room to spare based on actual FTE, not on perceived FTE.
- *Parallel Arrivals Under All Meteorological Conditions*
  - Closely-spaced parallel runways may be used for simultaneous arrival streams, rather than a long queue on a single runway, regardless of weather conditions.
- *Improved Runway Occupancy*
  - Aircraft that touchdown on the actual arrival runway centerline exit the runway more safely and quicker than those that must correct their position after touchdown.
- *Reducing Collision Risk*
  - As aircraft become bigger and wider, the need to keep aircraft safely laterally separated becomes more and more important.

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[2]Let $\chi^2(n)$ be a (central) chi-square distribution with n degrees of freedom. Then

$$P\left(\chi^2(n) \leq x\right) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \int_0^x u^{\frac{n}{2}-1} e^{-\frac{1}{2}u}\, du$$

so that

$$\begin{aligned} P\left(\sqrt{\chi^2(n)} \leq z\right) &= P\left(\chi^2(n) \leq z^2\right) \\ &= \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \int_0^{z^2} u^{\frac{n}{2}-1} e^{-\frac{1}{2}u}\, du \\ &= \frac{2}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \int_0^{z} t\left(t^2\right)^{\frac{n}{2}-1} e^{-\frac{1}{2}t^2}\, dt,\, u = t^2 \\ &= \frac{1}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} \int_0^{z} t^{n-1} e^{-\frac{1}{2}t^2}\, dt \end{aligned}$$

which means

$$P\left(\sqrt{\chi^2(n)} = z\right) = \frac{1}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} z^{n-1} e^{-\frac{1}{2}z^2}$$

When $n = 2$, we have

$$P\left(\sqrt{\chi^2(2)} = z\right) = ze^{-\frac{1}{2}z^2}$$

[3]The angle of rotation could also be $-\frac{\pi}{4} - \text{Arctan}\, m$ so that $m = -1$. In either case, $m^2 = 1$ so that $A = |B| = C = \frac{1}{2}$.

6. Enhancements

As with all analytical systems, improvements and modernization adjustments should be a standard part of its maintenance.

- Consider other "best fit" methods besides least-squares, such as ...
  - Weighted least-squares;
  - Bayesian prior on likelihood of a particular approach path;
  - Continuous descent arrivals.
- Calculate posterior distribution of FTE distances based on prior calculations as a function of the distance from the arrival runway threshold.
- Game theoretic approach to FTE-related risk management.
  - In a two player, zero-sum game of "safety" versus "nature," the actual FTE value could be the utility function.
- Sensor calibration based on hypothesized zero FTE.
  - Designed experiments could assess the component contributions of FTE.
- Develop FTE-analysis-enabled avionics implementations.

7. Summary

There are safety and efficiency reasons to minimize FTE for arrivals at any airport; however, it is especially important at busy airports, and ones with closely-spaced parallel runways.

Airborne avionics vary significantly from one aircraft to another, and ground-based and space-based surveillance systems often provide highly variable position, velocity, acceleration, and altitude information.

By considering position data as samples from a stochastic process, FTE may be assessed (and steps taken therefrom to minimize the FTE) by finding the least-squares perceived arrival runway extended centerline, which becomes the calculated actual arrival runway extended centerline, and calculating FTE relative to this reference line.

This memorandum provides the analytical methods for implementing an assessment FTE algorithm in embedded systems.

8. Appendix: Distance From A Point To A Line

Given two distinct points (x_1, y_1) and (x_2, y_2), the unique line joining these points is given by

$$L = \left\{ (x, y) : y = y_1 + \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \right\}$$

with the understanding that $L = \{(x, y) : x = x_1\}$ if $x_1 = x_2$. Note that $x_1 = x_2$ and $y_1 = y_2$ cannot occur at the same time as the points (x_1, y_1) and (x_2, y_2) are distinct.

The line perpendicular to L going through the point (x_0, y_0) is given by

$$L' = \left\{ (x, y) : y = y_0 - \frac{x_2 - x_1}{y_2 - y_1} (x - x_0) \right\}$$

The intersection of L and L' is the point (x_I, y_I) such that

$$y_0 - \frac{x_2 - x_1}{y_2 - y_1} (x_I - x_0) = y_1 + \frac{y_2 - y_1}{x_2 - x_1} (x_I - x_1)$$

or

$$x_I = \frac{y_0 - y_1 + \frac{x_2 - x_1}{y_2 - y_1} x_0 + \frac{y_2 - y_1}{x_2 - x_1} x_1}{\frac{x_2 - x_1}{y_2 - y_1} + \frac{y_2 - y_1}{x_2 - x_1}}$$

If $m = \frac{y_2 - y_1}{x_2 - x_1}$, then we have

$$\begin{aligned} x_I &= \frac{y_0 - y_1 + \frac{1}{m}x_0 + mx_1}{\frac{1}{m} + m} \\ &= \frac{x_1 m^2 + (y_0 - y_1)\, m + x_0}{m^2 + 1} \end{aligned}$$

and

$$\begin{aligned} y_I &= y_1 + m\left(\frac{(y_0 - y_1)\, m + m^2 x_1 + x_0}{m^2 + 1} - x_1\right) \\ &= \frac{y_0 m^2 + (x_0 - x_1)\, m + y_1}{m^2 + 1} \end{aligned}$$

so that

$$(x_I, y_I) = \left(\frac{x_1 m^2 + (y_0 - y_1)\, m + x_0}{m^2 + 1}, \frac{y_0 m^2 + (x_0 - x_1)\, m + y_1}{m^2 + 1}\right)$$

If $A = \frac{m^2}{m^2+1}$, $B = \frac{m}{m^2+1}$, and $C = \frac{1}{m^2+1}$ then

$$(x_I, y_I) = (Ax_1 + B\,(y_0 - y_1) + Cx_0, Ay_0 + B\,(x_0 - x_1) + Cy_1)$$

Then the distance from (x_0, y_0) to (x_I, y_I) is

$$D = \sqrt{(Ax_1 + B\,(y_0 - y_1) + (C - 1)\, x_0)^2 + ((A - 1)\, y_0 + B\,(x_0 - x_1) + Cy_1)^2}$$